

# A remark on singularities of primitive cohomology classes

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November 2007

## Abstract

Green and Griffiths have introduced several notions of singularities associated with normal functions, especially in connection with middle dimensional primitive Hodge classes. In this note, by using the more elementary aspects of the Decomposition Theorem, we define global and local singularities associated with primitive middle dimensional cohomology classes and by using the Relative Hard Lefschetz Theorem, we show that these singularities detect the global and local triviality of the primitive class. In a final section, we write-up a classical inductive argument relating the Hodge conjecture to the local non-vanishing of primitive classes.

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# 1 Introduction

In [7], Thomas has shown that the Hodge Conjecture (HC) is equivalent to the existence of special hyperplane sections; see §3.3. In [6], Green and Griffiths have introduced several notions of singularities of normal functions, especially in connection with middle dimensional primitive Hodge classes, and have tied these notions to a positive answer to the HC. Roughly speaking, the HC is equivalent to the non vanishing of these singularities at some point. As is well-known, by virtue of standard inductive arguments, the middle-dimensional case, is the critical one.

The various notions of singularities in [6] reflect the subtle geometries which one needs to explore to further this point of view. In this note, we concentrate on what is perhaps the most simple of these notions and propose an essentially elementary definition (cf. Definition 3.3) of what we call the Green-Griffiths singularity associated with a primitive middle dimensional class on a projective manifold. There are a local (i.e. having to do with a hyperplane section) and a global version. Our definition is possible in view of the most basic properties of the perverse filtration that can be read-off directly from the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber [1]. We prove, using the Relative Hard Lefschetz Theorem [1], Propositions 3.6 and 3.7 which establish that the local/global triviality of the primitive class is detected precisely by the Green-Griffiths invariants.

§2 is for preliminaries on the Decomposition Theorem. §3 contains the results of this note mentioned above. §4 contains what we believe to be a classical and well-known inductive argument concerning the HC. We could not find what we think is a complete reference for this result that puts into context the notion of singularity introduced by Green and Griffiths.

We have reached the conclusions contained in §3 in the Spring of 2007, during our visit at I.A.S., Princeton.

By using M. Saito's theories of mixed Hodge modules and admissible normal functions, the preprint [2] introduces, among other things, a notion of singularity that essentially coincides with ours. Their methods and ours are entirely different and we hope that the two different points of view can both be useful for further geometric investigations.

We thank Phil Griffiths for inspiring conversations.

## 2 Decomposition Theorem formulæ

In this section we collect the facts we need from the Decomposition Theorem in [1].

### 2.1 Set-up

We work with rational cohomology. Let:

- $X^{2n} \subseteq \mathbb{P}^d$  be a smooth, irreducible and projective manifold of dimension  $2n$ ;
- $\mathcal{L}$  be the hyperplane bundle;

- $\zeta \in H^{2n}(X)$  be a cohomology class;
- there is the universal hyperplane family (for simplicity, we write  $\mathbb{P}$  also for  $\mathbb{P}^\vee$ )

$$X \xleftarrow{q} \mathfrak{X} \xrightarrow{\pi} \mathbb{P}^d, \quad \dim \mathfrak{X} = 2n - 1 + d$$

and we have the hyperplane sections as fibers

$$\mathfrak{X}_p := \pi^{-1}(p), \quad \dim \mathfrak{X}_p = 2n - 1.$$

The main object of investigation here are the classes

$$q^*\zeta \in H^{2n}(\mathfrak{X}), \quad (q^*\zeta)|_{\mathfrak{X}_p} = \zeta|_{\mathfrak{X}_p} \in H^{2n}(\mathfrak{X}_p).$$

Let  $Z$  be a variety  $L$  be a local system on a dense open subset  $U \subseteq Z_{reg}$ . The intersection cohomology complex  $IC_Z(L)$  is a complex of sheaves on  $Z$ . Its cohomology sheaves satisfy

$$\mathcal{H}^l(IC_Z(L)) = 0, \quad l \neq [-\dim Z, -1]. \quad (1)$$

We have the intersection cohomology groups

$$IH^k(Z, L) := \mathbb{H}^{k-\dim Z}(Z, IC(L)). \quad (2)$$

Clearly,

$$IH^k(Z, L) = 0, \quad k \notin [0, 2 \dim Z]. \quad (3)$$

## 2.2 The Decomposition Theorem for $\pi$

The Decomposition Theorem for  $\pi : \mathfrak{X} \rightarrow \mathbb{P}^d$  gives a non canonical decomposition

$$\phi : \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} IC(L_{ij})[-i - (2n - 1 + d)] \simeq R\pi_* \mathbb{Q} \quad (4)$$

where  $L_{ij}$  is a local system on the codimension  $j$  stratum  $S_{d-j} \subseteq \mathbb{P}^d$ . Strata are not connected, so that by  $IC$  here we mean the direct sum of the  $IC$ 's on the connected components of the same dimension.

The  $L_{i0}$  are the local systems on  $S_d \subseteq \mathbb{P}^d \setminus X^\vee$  :

$$L_{i0} = R^{2n-1+i} := (R^{2n-1+i} \pi_* \mathbb{Q})|_{S_d}. \quad (5)$$

In what follows, the pedantic notation is to make the formulæ ready to use later. We have a non canonical decomposition for the cohomology groups

$$H^l(\mathfrak{X}) = \phi \left( \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} IH^{(l-2n)-j-(i-1)}(\overline{S_{d-j}}, L_{ij}) \right), \quad (6)$$

one for cohomology sheaves

$$R^l \pi_* \mathbb{Q} = \phi \left( \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(l-2n)-(d-j)-j-(i-1)}(IC(L_{ij})) \right) \quad (7)$$

and one for the cohomology groups

$$(R^l \pi_* \mathbb{Q})_p = H^l(\mathfrak{X}_p) = \phi \left( \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(l-2n)-(d-j)-j-(i-1)}(IC(L_{ij}))_p \right) \quad (8)$$

### 2.3 Definition of the filtrations on $H(\mathfrak{X})$ and $H(\mathfrak{X}_p)$

These filtrations are discussed and used in our paper [3]. The theory of perverse sheaves filters the groups  $H(\mathfrak{X})$ . The Decomposition Theorem makes this more visible.

**Remark 2.1 (Reminder on the perverse filtration)** The perverse filtration on  $H(\mathfrak{X})$  is by Hodge substructures and it coincides, up to some shifts, with the monodromy filtration associated with the nilpotent cup-product action of  $q^*L$  on  $H(\mathfrak{X})$  [3]. A splitting  $\phi_{\mathcal{L}}$  in the category of Hodge structures exists, as shown in [4], but we do not need it here (see Remark 3.5). Over the regular part of  $\pi$ , the perverse filtration coincides, up to re-numbering, with the filtration coming from the Leray spectral sequence.

The **perverse filtration** on  $H^l(\mathfrak{X})$  is the increasing filtration indexed by  $i \in \mathbb{Z}$  :

$$H_{\leq i}^l(\mathfrak{X}) := \phi \left( \bigoplus_{i' \leq i} \bigoplus_{j \in \mathbb{N}} IH^{(l-2n)-j-(i'-1)}(\overline{S_{d-j}}, L_{i'j}) \right) \subseteq H^l(\mathfrak{X}). \quad (9)$$

The perverse filtration is independent of the splitting  $\phi$ . The graded pieces  $H_i^l(\mathfrak{X})$  are canonically isomorphic to

$$H_i^l(\mathfrak{X}) = \bigoplus_{j \in \mathbb{N}} IH^{(l-2n)-j-(i-1)}(\overline{S_{d-j}}, L_{ij}). \quad (10)$$

The decomposition  $\phi$  induces, in the same way an increasing filtration on the stalks which we call the **induced filtration** (also independent of  $\phi$ )

$$H_{\leq i}^l(\mathfrak{X}_p) = \phi \left( \bigoplus_{i' \leq i} \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(l-2n)-(d-j)-j-(i'-1)}(IC(L_j)) \right)_p \subseteq H^l(\mathfrak{X}_p), \quad (11)$$

with graded pieces canonically isomorphic to

$$H_i^l(\mathfrak{X}_p) = \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(l-2n)-(d-j)-j-(i-1)}(IC(L_j))_p. \quad (12)$$

The restriction map  $r : H(\mathfrak{X}) \rightarrow H(\mathfrak{X}_p)$  is filtered and strict (obvious since the map is a direct sum map) with respect to the perverse and induced filtrations:

$$r : H_{\leq i}(\mathfrak{X}) \rightarrow H_{\leq i}(\mathfrak{X}_p).$$

If we fix a small neighborhood  $U \subseteq \mathbb{P}^d$  of  $p$  and set  $\mathfrak{X}_U := \pi^{-1}(U)$ , then we have filtered isomorphisms

$$r : H_{\leq i}(\mathfrak{X}_U) \simeq H_{\leq i}(\mathfrak{X}_p). \quad (13)$$

### 3 The Green-Griffiths singularity

#### 3.1 Bound on the filtrations on $H(\mathfrak{X})$ and $H(\mathfrak{X}_p)$ .

In order to define the Green-Griffiths singularity we need (ii) below.

**Lemma 3.1** (i)

$$H_{\leq 1}^{2n}(\mathfrak{X}) = H^{2n}(\mathfrak{X}), \quad H_{\leq 1}^{2n}(\mathfrak{X}_p) = H^{2n}(\mathfrak{X}_p).$$

(ii) If  $\zeta \in H^{2n}(X)$  is primitive, then

$$q^*\zeta \in H_{\leq 0}^{2n}(\mathfrak{X}), \quad \zeta|_{\mathfrak{X}_p} \in H_{\leq 0}^{2n}(\mathfrak{X}_p).$$

*Proof.* By virtue of the obvious vanishing (3) in negative degrees, the graded piece (10)

$$H_i^{2n}(\mathfrak{X}) = \bigoplus_{j \in \mathbb{N}} IH^{-j-(i-1)}(\overline{S_{d-j}}, L_{i,j}) = 0, \quad \forall i \geq 2. \quad (14)$$

This holds also for  $\mathfrak{X}_U$ . In view of (13), this proves (i).

The graded piece

$$H_1^{2n}(\mathfrak{X}) = \bigoplus_{j \in \mathbb{N}} IH^{-j}(\overline{S_{d-j}}, IC(L_{1,j})) = IH^0(\mathbb{P}^d, L_{10}).$$

By (5),  $L_{10} = R^{2n}$  is the local system on the dense stratum of  $\mathbb{P}^d$  corresponding to the variation of  $H^{2n}(\mathfrak{X}_\eta)$ , where  $\mathfrak{X}_\eta$  is a smooth hyperplane section. The group in question is just the space of global invariants. Since we are assuming that  $\zeta$  is primitive,  $\zeta|_{\mathfrak{X}_\eta} = 0$  defines the zero section in this group and (ii) follows.  $\square$

#### 3.2 Definition of the Green-Griffiths invariant

We have the decompositions: (10) for  $H(\mathfrak{X})$  (non canonical), (6) for the graded  $H_i(\mathfrak{X})$  (canonical), (12) for  $H(\mathfrak{X}_p)$  (non canonical) and (8) for the graded  $H_i(\mathfrak{X}_p)$  (canonical).

Let  $\zeta \in H^{2n}(X)$ . There is the non canonical decomposition associated with  $\phi$ :

$$q^*\zeta = \phi\left(\sum_{ij} [q^*\zeta]_{ij}\right), \quad \zeta|_{\mathfrak{X}_p} = \phi\left(\sum_{ij} [\zeta|_{\mathfrak{X}_p}]_{ij}\right), \quad (15)$$

where the terms  $[-]_{ij}$  depend on  $\phi$ .

Let  $\zeta \in H^{2n}(X)$  be primitive. Then, by Lemma 3.1, the terms  $[-]_0$  and  $[-]_{0j}$  are well-defined, independently of  $\phi$ :

$$[q^*\zeta]_0 = \sum_{j \in \mathbb{N}} [q^*\zeta]_{0j} \in H_0^{2n}(\mathfrak{X}) = \bigoplus_{j \in \mathbb{N}} IH^{-j+1}(\overline{S_{d-j}}, IC(L_{0j})),$$

$$[\zeta|_{\mathfrak{X}_p}]_0 = \sum_{j=0}^d [\zeta|_{\mathfrak{X}_p}]_{0j} \in H_0^{2n}(\mathfrak{X}_p) = \bigoplus_{j \in \mathbb{N}} \mathcal{H}^{-d+1}(IC(L_{0j}))_p.$$

As in the proof of Lemma 3.1, complemented by the support condition (1), the terms with  $j \geq 2$  are zero:

$$[q^*\zeta]_0 = [q^*\zeta]_{00} + [q^*\zeta]_{01}, \quad (16)$$

$$[\zeta]_{\mathfrak{x}_p}]_0 = [\zeta]_{\mathfrak{x}_p}]_{00} + [\zeta]_{\mathfrak{x}_p}]_{01} \quad (17)$$

where we write explicitly, remembering that our notation calls for  $IC_Z(L)$  to have cohomology sheaves in the interval  $[-\dim Z, -1]$ :

$$[\zeta]_{\mathfrak{x}_p}]_{00} \in \mathcal{H}^{-d+1}(IC(R^{2n-1}))_p, \quad [\zeta]_{\mathfrak{x}_p}]_{01} \in \mathcal{H}^{-d+1}(IC(L_{01}))_p. \quad (18)$$

**Remark 3.2** The local system  $L_{01}$  is usually defined on the regular part of the dual variety  $X^\vee \subseteq \mathbb{P}^d$ . If we take the embedding associated with  $m\mathcal{L}$ ,  $m \gg 0$ , then the local system  $L_{01} = 0$ . This follows from [SGA 7.2, XVIII. 5.3.5 (“Condition A”) and 6.4 (“Condition A” is verified for  $m \gg 0$ )]. In fact, the stalk of  $L_{01}$  at a general point of the dual hypersurface measures the failure of the adjunction map 5.3.2 (loc.cit.) to be an isomorphism (it is surjective for Lefschetz pencils). This can be also seen by using the Clemens-Schmid sequence.

**Definition 3.3** Let  $\zeta \in H^{2n}(X)$  be a primitive class.  
The global Green-Griffiths invariant  $s(\zeta)$  is defined to be

$$s(\zeta) := [\zeta]_{00} \in IH^1(\mathbb{P}^d, IC(R^{2n-1})).$$

The local Green-Griffiths invariant  $s(\zeta)_p$  is defined to be

$$s(\zeta)_p := [\zeta]_{\mathfrak{x}_p}]_{00} \in \mathcal{H}^{-d+1}(IC(R^{2n-1}))_p.$$

Clearly, these invariants depend on the embedding.

**Remark 3.4** From the conditions of support of  $IC$ , it follows that the locus

$$\text{Sing}(\zeta) := \{p \in \mathbb{P}^d \mid s(\zeta)_p \neq 0\}$$

is of codimension at least two.

**Remark 3.5** The Hodge-theoretic splitting  $\phi_{\mathcal{L}}$  of [4] shows that  $H_0^{2n}(\mathfrak{X})$  is endowed with a natural pure Hodge structure so that if  $\zeta$  is a Hodge class, then so are  $[\zeta]_0$  and  $s(\zeta)$ . Our paper [4] does not afford local results. However, using the M. Saito’s general theory of mixed Hodge modules one can reach similar conclusions for  $[\zeta]_{\mathfrak{x}_p}]_0$  and  $s(\zeta)_p$

### 3.3 R. Thomas’ result

Thomas [2005] has proved that the Hodge conjecture is equivalent, given an arbitrary middle dimensional primitive Hodge class  $\zeta$ , to the existence of  $m \gg 0$  such that there exists  $p \in |m\mathcal{L}|$  with  $\mathfrak{x}_p$  nodal and  $\zeta|_{\mathfrak{x}_p} \neq 0$ . If we drop the nodal requirement, the resulting weaker statement is of course still true. Because of primitivity, the hypersurface must be singular and nodality is an interesting improvement.

### 3.4 The classes $[q^*\zeta]_0$ and $[\zeta]_{|\mathfrak{X}_p}|_0$ detect global/local triviality

Given a primitive class  $\zeta \in H^{2n}(X)$ , the class  $q^*\zeta \in H_{\leq 0}^{2n}$  and it defines a canonical element  $[q^*\zeta]_0 \in H_0^{2n}(\mathfrak{X})$ . Ditto for  $\zeta|_{\mathfrak{X}_p} \in H_{\leq 0}^{2n}(\mathfrak{X}_p)$  and  $[\zeta]_{|\mathfrak{X}_p}|_0 \in H_0^{2n}(\mathfrak{X}_p)$ .

The following proposition ensures that, given *any* embedding  $|\mathcal{L}|$ , the global/local triviality of primitive classes is detected by the global/local classes  $[-]_0$ .

It is a simple consequence of the Relative Hard Lefschetz Theorem [1]. In particular, we only need that cupping with the relatively ample line bundle  $q^*\mathcal{L}$  is injective on  $H_{\leq -1}(\mathfrak{X})$  and on  $H_{\leq -1}(\mathfrak{X}_p)$  (see [4]).

**Proposition 3.6** *Let  $\zeta \in H^{2n}(X)$  be primitive.*

- (i) *The class  $\zeta = 0$  IFF  $[q^*\zeta]_0 = 0$ .*
- (ii) *The class  $\zeta|_{\mathfrak{X}_p} = 0$  IFF  $[\zeta]_{|\mathfrak{X}_p}|_0 = 0$ .*

*Proof.* We prove (i). The proof for (ii) is analogous. One direction is trivial. Let  $\zeta \neq 0$ , so that  $q^*\zeta \neq 0$ . If  $[q^*\zeta]_0 = 0$ , then  $q^*\zeta \in H_{\leq -1}^{2n}(\mathfrak{X})$ . By the Relative Hard Lefschetz, the cup product with  $q^*\mathcal{L}$  is injective on  $H_{\leq -1}^{2n}(\mathfrak{X})$ , contradicting  $q^*(\mathcal{L} \cdot \zeta) = 0$ .  $\square$

### 3.5 The local Green-Griffiths invariants detects global/local triviality

The Green-Griffiths invariant captures the primitive class, i.e. we have the following, where by  $m \gg 0$ , we mean that we replace the embedding given by  $|\mathcal{L}|$  with the one given by  $|m\mathcal{L}|$ , with  $m \gg 0$ :

**Proposition 3.7** *Let  $\zeta \in H^{2n}(X)$  be primitive.*

- (i) *The class  $\zeta = 0$  IFF  $s(\zeta) = 0$ .*
- (ii) *Let  $m \gg 0$ . The class  $\zeta|_{\mathfrak{X}_p} = 0$  IFF  $s(\zeta)_p = 0$ .*

*Proof.* By Remark 3.2 and by (16) and (17), we have that, for  $m \gg 0$ :

$$s(\zeta)_p = [\zeta]_{|\mathfrak{X}_p}|_{00} = [\zeta]_{|\mathfrak{X}_p}|_0$$

and we apply Proposition 3.6 to deduce (ii). The proof of (i) is identical (except for the fact that the result holds for every embedding).  $\square$

**Remark 3.8** Proposition 3.7 implies that one can state R. Thomas' result using  $[\zeta]_{|\mathfrak{X}_p}|_0$ , or  $s(\zeta)_p$ , instead of  $\zeta|_{\mathfrak{X}_p}$ . We do not need the Hodge-theoretic nature of  $\phi_{\mathcal{L}}$  to prove the resulting statement. One simply uses  $\zeta|_{\mathfrak{X}_p}$  and Deligne's mixed Hodge structures.

**Remark 3.9** If we understand correctly, the content of §6 of the paper [2] by Brosnan et al. is equivalent to Remark 3.8.

### 3.6 Further characterizations of vanishing of $s(\zeta)_p$

In this section we clarify the relation between  $s(\zeta)_p$  and two other invariants associated with  $\zeta$ .

The natural map  $\mathbb{Q}_{\mathfrak{X}_p} \rightarrow IC_{\mathfrak{X}_p}$  induces a map  $H^{2n}(\mathfrak{X}_p) \rightarrow IH^{2n}(\mathfrak{X}_p)$ . By [4], Thm. 3.2.1, the kernel of this map is precisely  $W_{2n-1}H^{2n}(\mathfrak{X}_p)$ . Since  $\zeta|_{\mathfrak{X}_p}$  is of type  $(n, n)$  for the mixed Hodge structure, it is not in the kernel and we have the following (cf. [6], Thm. 2.ii):

**Corollary 3.10** Let  $\zeta \in H_{\mathbb{Q}}^{n,n}(X)$  be a primitive Hodge class on  $X$ .

The class  $[\zeta|_{\mathfrak{X}_p}]_0 = 0$  IFF  $\zeta|_{\mathfrak{X}_p} = 0$  in  $IH^{2n}(\mathfrak{X}_p) = IH_{2n-2}(\mathfrak{X}_p)$ .

Let  $m \gg 0$ . The Green-Griffiths invariant  $s(\zeta)_p = 0$  IFF  $\zeta|_{\mathfrak{X}_p} = 0$  in  $IH^{2n}(\mathfrak{X}_p) = IH_{2n-2}(\mathfrak{X}_p)$ .

It is possible to give a more precise characterization for the locus in which the Green Griffiths invariant does not vanish, which appears as a natural generalization of the condition for the (non) extendability of a normal function in [5]. We need the preliminary

**Lemma 3.11** Let  $U$  be a contractible neighborhood of  $0 \in \mathbb{C}^d$ , let  $D \ni 0$  be a divisor, and  $L$  be a local system on  $U \setminus D$ . Then  $\mathcal{H}^{1-d}(IC(L))_0$  injects naturally in  $\mathbb{H}^1(U \setminus D, L)$ .

*Proof.* Let  $U^* = U_d \subseteq U_{d-1} \subseteq \dots U_0 = U$  be the ascending chain of open subsets  $U_l = \coprod_{i \geq l} S_i$  associated with a stratification of  $(\mathbb{C}^d, D)$ , and denote by  $j_l : U_{l+1} \rightarrow U_l$  the corresponding imbeddings. We have the well-known formula

$$IC_U(L) := \tau_{\leq -1} Rj_{0*}(\dots (\tau_{\leq -d+1} Rj_{d-2*}(\tau_{\leq -d} Rj_{d-1*} L[d])) \dots).$$

Since the truncations relative to  $j_i$  for  $i \leq d-2$  are in degree bigger than or equal to  $1-d$ , and  $\tau_{\leq -d} Rj_{d-1*} L[d] = (j_{d-1*} L)[d]$ , setting  $J : U_{d-1} \rightarrow U$  we have

$$\mathcal{H}^{1-d}(IC(L))_0 = \mathbb{H}^{1-d}(U, RJ_* j_{d-1*} L[d]) = H^1(U_{d-1}, j_{d-1*} L).$$

The latter cohomology group is the term  $E_2^{10}$  in the Grothendieck spectral sequence for  $\mathbb{H}^1(U_{d-1}, RJ_{d-1*} L) = H^1(U^*, L)$ . The statement follows from the edge sequence.  $\square$

Recall our notation  $\mathfrak{X}_U = \pi^{-1}(U)$ .

**Corollary 3.12** Let  $U$  be a contractible neighborhood of  $p$ , and  $U^* = U \setminus U \cap X^\vee$ . Let  $m \gg 0$ . Then  $s(\zeta)_p = 0$  IFF  $q^* \zeta|_{\mathfrak{X}_{U^*}} = 0$  in  $H^{2n}(\mathfrak{X}_{U^*})$ .

*Proof.* Let  $u : U^* \rightarrow U$  be the open imedding. The map  $\pi : \pi^{-1}(U^*) \rightarrow U^*$  is smooth so that we have, by Deligne's theorem, the splitting on the right end side below:

$$R\pi_* \mathbb{Q}_{\mathfrak{X}_U} \longrightarrow Ru_* u^* R\pi_* \mathbb{Q}_{\mathfrak{X}_U} \simeq \oplus Ru_* R^l \pi_* \mathbb{Q}_{\mathfrak{X}_{U^*}}[-l].$$

The statement follows from 3.7.ii and 3.11 applied to  $L = R^{2n-1} = R^{2n-1} \pi_* \mathbb{Q}_{\mathfrak{X}_{U^*}}$ .  $\square$



## 4 Relation between HC and $s(\zeta)_p$

The content of this section is well-known and is essentially a re-writing of an argument of R. Thomas to be found in [7]. We do not fully understand Thomas's Hilbert scheme argument: we are un-able to rule out the existence of parasite contributions in the cycle he constructs that do not ensure that it restricts to the hyperplane sections as wanted.

Let us summarize briefly what follows. If the HC holds, after passing to  $|m\mathcal{L}|$ ,  $m \gg 0$ , a primitive Hodge class  $\zeta$  has  $\zeta|_{\mathfrak{X}_p} \neq 0$  for some point  $p \in \mathbb{P}^{d_m}$ . In fact, assuming HC, we find an effective cycle pairing non-trivially with  $\zeta$ . Then we find a hypersurface  $V \in |m\mathcal{L}|$ ,  $m \gg 0$ , containing that cycle. Then  $\zeta|_V \neq 0$ . This  $V$  gives the wanted  $p \in \mathbb{P}^{d_m}$ .

Conversely, using an inductive argument, if  $\zeta|_{\mathfrak{X}_p} \neq 0$  for some  $p \in \mathbb{P}^d$ , then one produces an algebraic  $n$ -cycle in  $\mathfrak{X}_p$  that pairs non trivially with  $\zeta$ . This means that the arbitrary, primitive Hodge class  $\zeta$  is not perpendicular to algebraic cycles and this implies HC.

What follows is presumably classic and well-known.

### 4.1 HC and $A^{k\perp}$

Let  $Y$  be a smooth projective manifold of dimension  $d$ . Fix  $k \in \mathbb{Z}$ . We have

$$A^k(Y) \subseteq H_g^k(Y) \subseteq H^{k,k}(Y) \subseteq H_{\mathbb{C}}^{2k}(Y),$$

with  $H_g^k(Y)$  the Hodge classes i.e. the pure Hodge substructure given by  $H_g^k(Y) = H^{k,k}(Y) \cap H_{\mathbb{Q}}^{2k}(Y) \subseteq H_{\mathbb{Q}}^{2k}(Y)$ , and with  $A^k(Y)$  the algebraic classes, i.e. the image of the cycle class map. By Hard Lefschetz, there are isomorphisms

$$H_g^k(Y) \simeq H_g^{d-k}(Y).$$

There is the cup product bilinear map

$$H_g^k(Y) \times H_g^{d-k}(Y) \longrightarrow \mathbb{Q}, \quad (a, b) \mapsto \int_Y a \cup b$$

This bilinear map is nondegenerate. Assume that  $2k \leq d$ , else  $2(d-k) \leq d$  and we can switch  $a$  with  $b$  in what follows. Since  $\dim_{\mathbb{Q}} H_g^k(Y) = \dim_{\mathbb{Q}} H_g^{d-k}(Y)$ , it is enough to show that if  $a \neq 0$ , then  $\int_Y a \cup -$  is not the zero map. This follows from the primitive decomposition and the Hodge Riemann bilinear relations.

We have the following immediate

**Fact 4.1** *The Hodge conjecture HC for  $Y$  is equivalent to an orthogonality statement:*

$$A^k(Y) = H_g^k(Y) \quad \text{IFF} \quad 0 = (A^k(Y))^{\perp} \subseteq H_g^{d-k}(Y).$$

**Lemma 4.2** *The following are equivalent:*

- 1) HC holds;
- 2) for every smooth projective even dimensional  $X^{2n}$ , we have  $A^n(X)^{\perp} = 0$ .

*Proof.* 1)  $\implies$  2): it is Fact 4.1.

2)  $\implies$  1). We fix  $Y^d$  and  $k \in \mathbb{Z}$ . We want to prove that  $A^{d-k}(Y) = H_g^{d-k}(Y)$ . By Fact 4.1, this is equivalent to showing that  $A^{d-k}(Y)^\perp = 0$ . Let  $a \in A^{d-k}(Y)^\perp$ . We need to show that  $a = 0$ . If  $d = 2k$ , then we are done by assumption 2). We have two cases:  $d < 2k$  and  $d > 2k$ . By contradiction, assume  $a \neq 0$ .

Assume  $d < 2k$ . Take  $X^{2k} := Y^d \times T^{2k-d}$ ,  $T$  any projective manifold of dimension  $2k - d$  and let  $p : X \rightarrow Y$  be the projection. Note that  $p^*$  is injective so that  $0 \neq p^*a \in H_g^k(X)$ . Let  $Z \in A^k(X)$ . Since, by the projection formula and the choice of  $a$ , we have that  $p^*a \cdot Z = a \cdot p_*Z = 0$ , we see that  $p^*a \in A^k(X)^\perp$ . By 2),  $p^*a = 0$  and this is a contradiction. Assume  $d > 2k$ . Slice down  $Y^d$  to  $X^{2k}$  by taking the complete intersection of  $d - 2k$  general hyperplane sections. Weak Lefschetz implies that  $0 \neq a|_X \in H_g^k(X)$ . By 2), we have that there must be an algebraic cycle  $Z$  on  $X$  such that  $a|_X \cdot Z \neq 0$ . This implies that  $a \cdot Z \neq 0$  (the product is now on  $Y$ ). This contradicts the assumption  $a \in A^{d-k}(Y)^\perp$ .  $\square$

## 4.2 HC and $\zeta|_{\mathfrak{X}_p}$

**Lemma 4.3** *Let  $X^{2n}$  be smooth and projective,  $\zeta \in H_g^{2n}$ . Assume that  $HC(X)$  holds. Then there are  $m \gg 0$ , and  $p \in \mathbb{P}^{d_m} = |m\mathcal{L}|$  such that*

$$\zeta|_{\mathfrak{X}_p} \neq 0.$$

*If, in addition,  $\zeta$  is primitive, then the Green-Griffiths invariant*

$$s(\zeta)_p \neq 0.$$

*Proof.* The pairing  $H_g^n(X) \times H_g^n(X) \rightarrow \mathbb{Q}$  is perfect. It follows that there is  $a \in H_g^n(X)$  such that  $\zeta \cdot a \neq 0$ . By HC,  $a = \sum r_i Z_i$ ,  $Z_i$  cycle classes. It follows that there is an irreducible codimension  $n$  cycle  $Z$  with  $\zeta \cdot Z \neq 0$ . This means that  $\zeta|_Z \in H^{2n}(Z) = H_{2n}(Z)^\vee$  is non-zero. By Serre, there is  $m \gg 0$  such that there is a, necessarily singular, element  $\mathfrak{X}_p$  of  $|m\mathcal{L}|$  containing  $Z$ . It follows that  $\zeta|_{\mathfrak{X}_p} \neq 0$ . We conclude by Proposition 3.7.  $\square$

**Remark 4.4** If  $\zeta$  is not primitive, then  $\zeta|_{\mathfrak{X}_p} \neq 0$  for general  $p$ . The point is to realize this for primitive  $\zeta$ , for which the above restriction is zero for general  $p$  and every  $m$ .

**Proposition 4.5** *Assume that for every  $X^{2n}$ ,  $0 \neq \zeta \in H_g^n(X)$  there is  $m \gg 0$  and  $p \in |m\mathcal{L}|$  with  $[\zeta|_{\mathfrak{X}_p}] \neq 0$ . Then HC holds for projective manifolds.*

*Proof.* We want to prove  $HC(d)$  for every  $d$ . The case  $d = 0$  is trivial. Assume we have done the case  $d - 1$ .

By taking a smooth hyperplane section we deal with  $H^k(X)$ ,  $k \geq d + 1$ .

Using a Lefschetz pencil we deal with  $H^k(\tilde{X})$ ,  $k \leq d - 1$ . Here is how. Take a nice Lefschetz pencil (condition A of SGA is ok)  $X \xleftarrow{u} \tilde{X} \rightarrow \mathbb{P}^1$ . By the blow-up formula, HC holds for

$X$  IFF it holds for  $\tilde{X}$ . So we work on  $H^k(\tilde{X})$ . Let  $v : X_t \rightarrow \tilde{X}$  be the general member. We get an isomorphism of pure Hodge structures

$$H^{k-2}(X_t)(-1) \oplus H^k(X) \xrightarrow{g_*, u^*} H^k(\tilde{X}) \quad (19)$$

$(H^j(X) = H^j(X_t)^{\pi_1})$ . By induction we have HC for  $X_t$  and  $g_*$  sends cycle classes to cycle classes. Let  $a \in H_g^k(X)$ . Then  $u^*a \in H_g^k(\tilde{X})$  and  $u^*a|_{X_t} \in H_g^k(X_t)$ . By HC for each smooth  $X_t$ , we have

$$N_1 u^*a|_{X_t} = N_3 Z'_t - N_2 Z''_t$$

with  $N_i \in \mathbb{N}^*$  and the  $Z'_t$ 's are effective cycle classes on  $X_t$ . For  $l \gg 0$ , we have that  $Z''_t$  is contained in a hypersurface of  $|l\mathcal{L}|$ . By iterating, we have that  $Z''_t$  is contained in the complete intersection of  $k/2$  hypersurfaces in  $|l\mathcal{L}|$ ,  $l \gg 0$ . It follows that

$$Z''_t = \nu \mathcal{L}^{k/2} - T_t$$

with  $\nu \gg 0$  and  $T_t$  effective on  $X_t$ . It follows that

$$N_1 u^*a|_{X_t} + N_2 \nu \mathcal{L}^{k/2} = N_3 Z'_t + N_2 T_t = \mathcal{Z}_t,$$

where  $\mathcal{Z}_t$  is an effective cycle (class). The set of  $t$ 's is uncountable. Take  $\text{Hilb}(\tilde{X}/\mathbb{P}^1)$ . It has a countable number of components. It follows that there is an irreducible component of it that maps onto  $\mathbb{P}^1$  and that it contains an uncountable number of the  $Z_t$ 's (which then share necessarily the  $N_1$  and  $N_2$ ). This irreducible component contains a curve  $C$  mapping surjectively to  $\mathbb{P}^1$  and passing through a point corresponding to one of the  $\mathcal{Z}_t$ . We take that  $t$  and that  $X_t$ . Take the base change family over  $C$ . This gives an algebraic cycle  $\tilde{\mathcal{Z}}'$  on  $\tilde{X}$  whose restriction to  $X_t$   $\mathcal{Z}'_t$  has cohomology class a positive  $N$  multiple of  $\mathcal{Z}_t$ . By (19)

$$\mathcal{Z}' = g_*\alpha + u^*\beta, \quad \mathcal{Z}'|_{X_t} = N\mathcal{Z}_t = NN_1 u^*a|_{X_t} + NN_2 \nu u^*\mathcal{L}^{k/2}, \quad g_*\alpha|_{X_t} = 0.$$

Since  $u^*$  is injective in this range by Weak Lefschetz,

$$\beta = NN_1 a + NN_2 \nu \mathcal{L}^{k/2}.$$

So, on  $\tilde{X}$  :

$$\mathcal{Z}' = g_*\alpha + u^*(NN_1 a + NN_2 \nu \mathcal{L}^{k/2}).$$

The summand  $g_*\alpha$  is algebraic by induction. It follows that  $u^*a$  is algebraic and we are done: we have dealt using induction with  $H_g^k(X)$ ,  $k < d - 1$ .

The argument above looks like the one in [7], however it is different. We felt the need to write this argument up, since [7] does not seem to deal with the possible parasites extra components that may arise in that construction after one restricts to a hyperplane section of the pencil.

If  $d$  is odd, then we are done. If  $d$  is even, the remaining case is  $H^{2n}(X^{2n})$ . We deal with this case. By the inductive hypothesis, we know that  $\text{HC}(2n - 1)$  is ok. By Lemma 4.2

it is enough to show that for every  $X^{2n}$  we have that  $A^n(X)^\perp = 0$ , i.e. that a non zero Hodge class  $\zeta \in H_g^n(X)$  cannot be perpendicular to algebraic cycles.

We have  $m \gg 0$  and  $p \in |m\mathcal{L}|$  with  $\zeta|_{\mathfrak{X}_p} \neq 0$ .

Take a desingularization  $f : \widetilde{\mathfrak{X}}_p \rightarrow \mathfrak{X}_p$ . Note that the domain may be disconnected. By mixed Hodge theory,  $H^{2n}(\mathfrak{X}_p) \rightarrow H^{2n}(\widetilde{\mathfrak{X}}_p)$  has as kernel the classes of weights  $\leq 2n - 1$ . In particular,  $f^*\zeta \neq 0 \in H_g^n(\widetilde{\mathfrak{X}}_p)$ .

By the inductive hypothesis HC(2n-1), the HC holds on  $\widetilde{\mathfrak{X}}_p$ . By Fact 4.1, there is a cycle class  $W \in A^{n-1}(\widetilde{\mathfrak{X}}_p)$  such that  $g^*\zeta \cdot W \neq 0$ , where  $g : \widetilde{\mathfrak{X}}_p \rightarrow X$ . It follows that  $\zeta \cdot g_*W \neq 0$ , i.e. we found a cycle class which is not perpendicular to  $\zeta$ .  $\square$

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